

Summer Assignment 2022

Vectors

A key requirement in physics is understanding vectors. Most students will have had a minimal if any exposure to vectors, therefore this summer assignment includes a vectors tutorial and practice problems. **Your completed assignment must be mailed to Physics Department, c/o Randall K. Cooper High School, with a postmark of not later than July 31, 2022.** To reduce postage costs, you are not required to mail back the entire packet; at a minimum, mail back any page that contains your answers and/or shows your work.

Do not wait until the week before the due date to begin this part of the summer assignment!

I suggest keeping a copy of your completed work before mailing it in—there will be a test on vectors early in the school year and the completed packet would make a great study guide.

The following are criteria for an acceptable submission of the summer assignment:

- Answers are hand-printed legibly by you, the student, in the designated places on the page.
- Use black #2 pencil (blue ink or black ink is okay) except where noted.
- Print your name at the top of the first page.
- Show your work in the space provided or on the back side of the paper, or on a piece of notebook paper. Don't make me guess how you got your answer!
- Mail the original of your work. Do not mail a photocopy.
- Sign and date the signature block at the bottom of the first page.

It might be a good idea to make a photocopy of the pages first, then work out the problems on the photocopy. Then copy your work neatly onto the original for mailing.

Since vectors involve drawing straight lines, it might be a good idea to invest in a straightedge.

The summer assignment is to be **your own work**. It is not group work, nor is it a partner assignment. By signing below, in ink, you agree that the work you submit is your own work, and that you completed it without the assistance—direct or indirect—of any other person, including persons who did this summer assignment in a previous year, or at another school. You may use printed or online materials as references. Furthermore, you agree that you did not assist any other AP Physics student in completing his/her AP Physics summer assignment, nor will you help any future AP Physics student in completing his/her AP Physics summer assignment. **Parent, by signing below, in ink, you acknowledge this summer assignment is your student's own work.**

Student Signature: _____

Date: _____

Parent Signature: _____

Date: _____

Vectors

Many of your science classes thus far in your academic career have involved basic mathematical skills such as addition, subtraction, multiplication, division, and the occasional logarithm, exponentiation, raising to a power, and solving for a particular variable. Additionally, you have had to graph data.

In precalculus math classes, you used trigonometric functions and their inverses, used polar and rectangular coordinates, graphed functions, and used linear algebra to solve systems of equations. I imagine you might have even used imaginary numbers.

The good news is that all those previous classes will come together in the practical realm of physics!

One of the concepts used extensively in physics is the vector. **A vector is simply a physical quantity that is comprised of both a magnitude and a direction.** For example, if someone walks 0.55 km due north from some starting point, the “magnitude” of the walk is 0.55 km and the “direction” of the walk is north. If someone starts from the same starting point, and walks east for 0.55 km, she will not end up at the same place as the previous person did. Likewise, if a person starts at the same starting point and walks 0.30 km due north, he will end up at a different spot than the first two walkers. **For two vectors to be equal, their magnitudes must be equal, and their directions must be equal.** (This simple example does not account for the curvature of the earth’s surface, and that’s okay for now.)

In contrast to vectors, most of the quantities you have worked with up until now are known as *scalars* (rhymes with “jailers”). Scalars do **not** have a direction, although some may have positive or negative values. Examples of scalar quantities include mass, volume, temperature, energy, time, wavelength, frequency, intensity, moles, and bank account balances.

Many quantities in math and science are often given symbols as placeholders. For example, in algebra, the letter “ x ” often represents an unknown scalar quantity. When an unknown quantity is a vector, we write it in a special way, with an arrow over it, like this: “ \vec{x} ”. Some textbooks instead use a boldface font to denote vector variables, such as “ \mathbf{x} .”

When describing a vector quantity, the “direction” component is usually expressed with respect to a coordinate system. For example, we described the walkers of the previous discussion using local cardinal directions of north, east, south, and west.

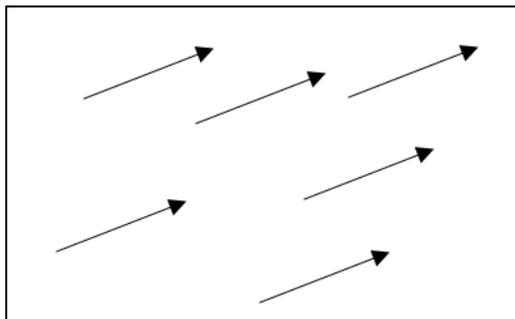
To depict a vector graphically, draw an arrow. The length of the arrow is proportional to the magnitude of the vector, and the direction of the arrow represents the direction of the vector. The following table provides examples, using a crude coordinate system based on the piece of paper, and lengths scaled as 1:1.

 2 cm, up	 3 cm, down	 2 cm, right	 2 cm, right & up
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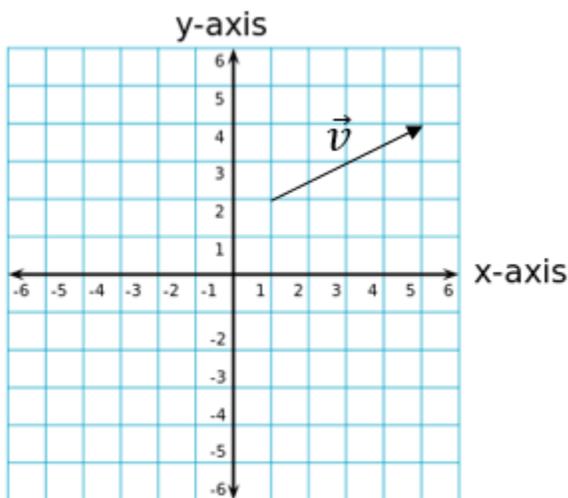
One can quickly see that it could become awkward describing vectors that do not align precisely with the directions established by the coordinate system. To that end, physics usually defines a coordinate frame of reference established by orthogonal (pronounced or-THOG-uh-null) axes. Orthogonal means mutually perpendicular. For

two-dimensional problems, the axes are usually called the x-axis and the y-axis. For three-dimensional problems, a z-axis is included. For now, we'll stick to 2-dimensional vectors, i.e., vectors that lie in a plane.

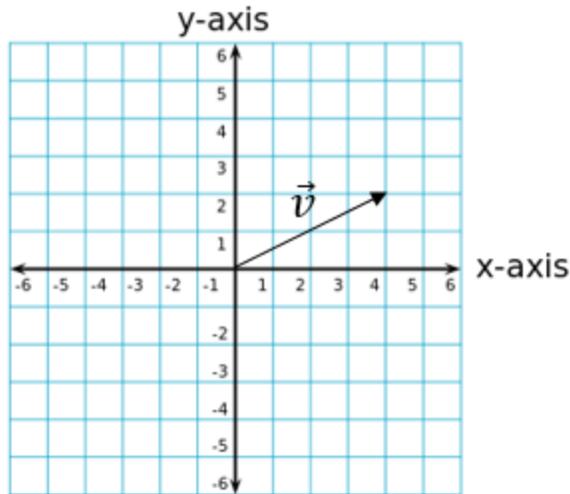
It is worth noting here that vectors are independent of their location in space. (The exception is a special vector called a "position vector" which—by definition—starts at the origin. When we say "starts at the origin" we mean that the "tail" of the arrow is at the origin of the coordinate system.) All the vectors represented in the following figure are equal, i.e., they all have the same magnitude and the same direction.



Getting back to establishing a coordinate system, this typically entails drawing an x-axis and a y-axis, perpendicular to each other, with the scaling the same on both axes. For example, one can draw a vector \vec{v} on an x-y coordinate system like this:



Don't let the fact that the vector doesn't start at the origin confuse you! To make this example easier to talk about, however, we can move the graphical representation of the vector so that it begins at the origin, as long as we don't change the direction or length of the vector. The above representation is equivalent to the following one:

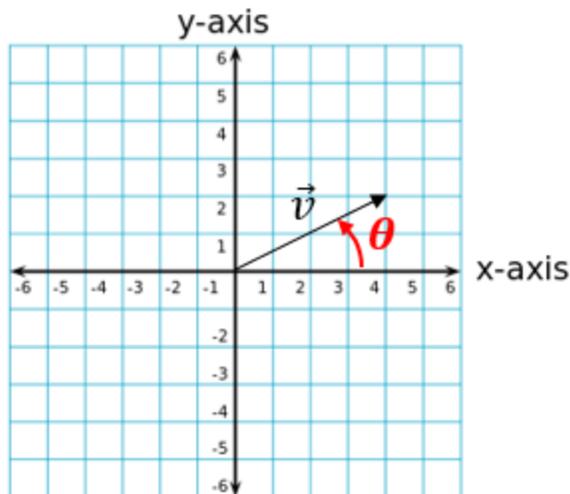


To describe this vector, we need some way to express its magnitude and its direction. The magnitude can be obtained from the Pythagorean Theorem:

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5} = 4.472$$

To denote magnitude, we use absolute value bars, or we omit the arrow over the vector symbol. Use of x- and y-subscripts denotes the x-component of the vector, and the y-component of the vector, respectively.

The direction of the vector can be obtained using trigonometry. **By standard convention, the angle of a vector is defined with respect to the positive half of the x-axis, with counterclockwise angle corresponding to the direction of positive angle.** Using the Greek letter theta (θ), the angle for the above vector can be drawn onto the graph like this:



The value for angle θ can be obtained by noting that the tangent of θ is simply the y-component of the vector divided by the x-component:

$$\tan \theta = \frac{v_y}{v_x} = \frac{2}{4}$$

$$\theta = \tan^{-1}\left(\frac{2}{4}\right) = 26.57^\circ$$

Thus, we could state what is essentially a “polar” representation of the vector \vec{v} :

$$\vec{v} = 4.472 \angle 26.57^\circ$$

The angle thus presented is assumed to be as measured counterclockwise from positive x-axis unless stated otherwise. If this vector were a velocity vector, and if the units on the graph were meters per second, then it is necessary to include the proper units in the expression for the vector:

$$\vec{v} = 4.472 \text{ m/s} \angle 26.57^\circ$$

Another common way of expressing the angle is to use the “@” symbol thus:

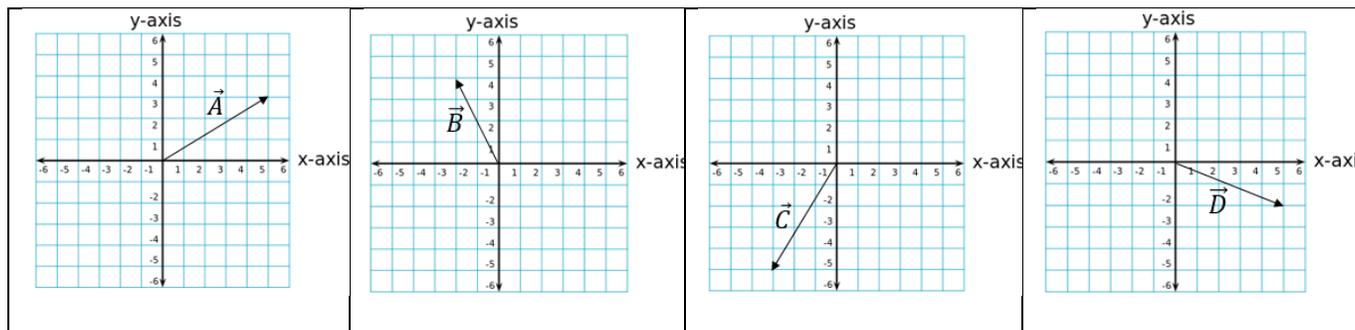
$$\vec{v} = 4.472 \text{ m/s} @ 26.57^\circ$$

Be cautious when using the inverse tangent function on your electronic calculators to calculate vector angles! If the vector (positioned with its tail at the origin) points into the first quadrant, the “calculator” answer for θ will be correct as is. But if the vector points into the second or third quadrant, the “calculator” value for theta needs to be corrected by adding 180 degrees to the calculator answer. If the vector points into the fourth quadrant, it can be corrected to a positive value by adding 360 degrees, or it can be left expressed as a negative value, with the understanding that it is measured *clockwise*. The following table illustrates this correction. **Use your own calculator to confirm these results.**

Vector	Graph	Quadrant	Calculation	Calculator Result	Correction	Corrected Result
$v_x = 4$ $v_y = 2$		I	$\theta = \tan^{-1}\left(\frac{2}{4}\right)$	$\theta_c = 26.57^\circ$	$\theta = \theta_c$	$\theta = 26.57^\circ$
$v_x = -4$ $v_y = 2$		II	$\theta = \tan^{-1}\left(\frac{2}{-4}\right)$	$\theta_c = -26.57^\circ$	$\theta = \theta_c + 180^\circ$	$\theta = 153.43^\circ$
$v_x = -4$ $v_y = -2$		III	$\theta = \tan^{-1}\left(\frac{-2}{-4}\right)$	$\theta_c = 26.57^\circ$	$\theta = \theta_c + 180^\circ$	$\theta = 206.57^\circ$
$v_x = 4$ $v_y = -2$		IV	$\theta = \tan^{-1}\left(\frac{-2}{4}\right)$	$\theta_c = -26.57^\circ$	$\theta = \theta_c + 360^\circ$ $(\theta = \theta_c)$	$\theta = 333.43^\circ$ $(\theta = -26.57^\circ)$

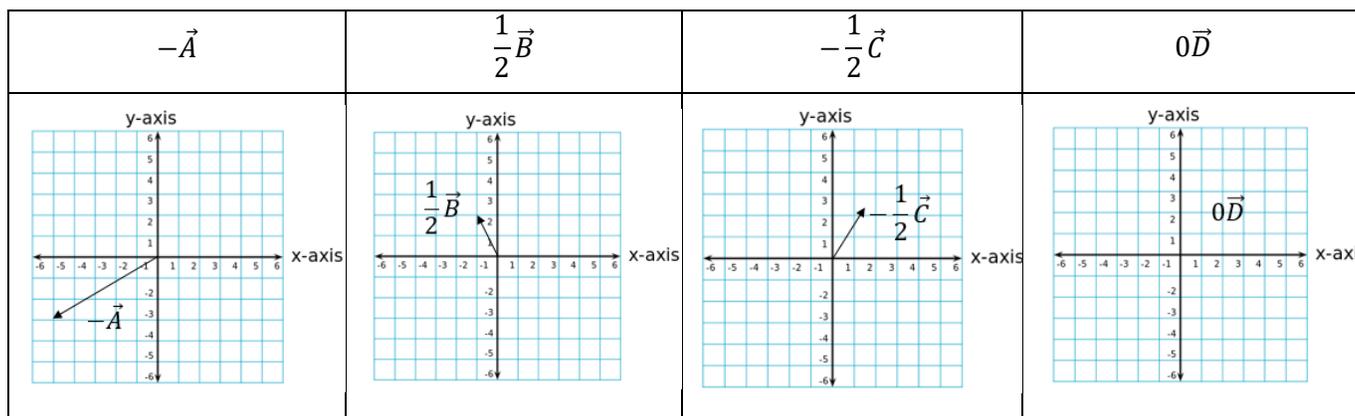
When doing this kind of calculation, it is worthwhile to quickly sketch the vector to check the feasibility of your calculator answer.

For practice, write a similar expression for each of the following vectors. The first one is already done for you. Don't worry about significant figures for the time being.

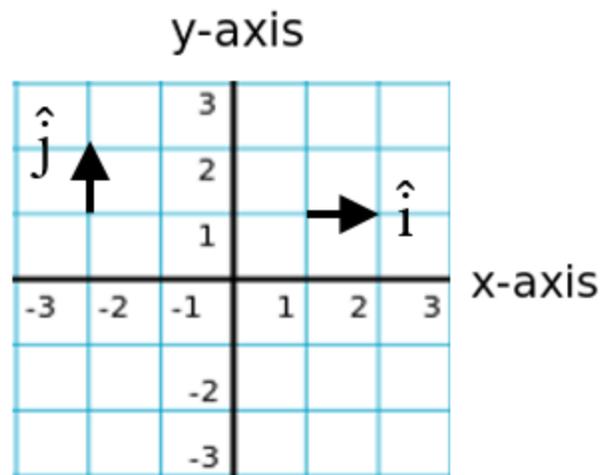


$\vec{A} = 5.831 @ 30.964^\circ$	$\vec{B} =$	$\vec{C} =$	$\vec{D} =$
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Multiplying a vector by a positive scalar changes the magnitude of the vector but not the direction. Multiplying a vector by a negative scalar can change the magnitude of the vector but it also changes the direction by 180 degrees. As an example, the four vectors sketched above can be multiplied by various scalars as shown below.



You might be wondering how to express vectors using *rectangular* coordinates. The “ordered pair” method you might have learned in previous math classes is replaced in physics by **unit vector** notation. **A unit vector is a vector of magnitude one (1) and a direction aligned with a coordinate system axis.** Instead of an arrow over the variable symbol, a unit vector symbol has a caret (^) over it. By standard convention, the unit vectors aligned with the x-, y-, and z-axes are called \hat{i} , \hat{j} , and \hat{k} , respectively. These unit vectors are pronounced “eye hat,” “jay hat,” and “kay hat,” respectively. The graph below shows example unit vectors for the x and y directions. The unit vector for the z direction points up out of the paper and is not shown. Remember that the location of a vector is irrelevant—only the magnitude and direction matter.



Any vector can be expressed as the sum of unit vectors, each of which may be multiplied by a scalar.

For practice, return to our original four example vectors and express them in unit-vector notation, also called rectangular notation and sometimes called Cartesian notation. The first one has been done for you:

$\vec{A} = 5\hat{i} + 3\hat{j}$	$\vec{B} =$	$\vec{C} =$	$\vec{D} =$

As a final note, some textbooks (especially math books) use a “column” notation for vectors. High school physics tends not to use the column notation, but in case you run across it in other textbooks, here’s what it would look like for our example vectors:

$\vec{A} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$	$\vec{B} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$	$\vec{C} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$	$\vec{D} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$
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To make matters even more confusing, some math textbooks—to save vertical space on the page—use “transpose notation” to express vectors:

$\vec{A} = [5 \ 3]^T$	$\vec{B} = [-2 \ 4]^T$	$\vec{C} = [-3 \ -5]^T$	$\vec{D} = [5 \ -2]^T$
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Vector Addition and Subtraction

The result of adding two or more vectors together is called the *resultant vector*, or simply the *resultant*. Two-dimensional vectors (those having only x- and y-components, for example) can be added graphically, with the accuracy of the result dependent on the skill of the person doing the drawing. There are two methods of graphically adding two vectors: the tip-to-tail method, and the parallelogram method. In both cases, a straightedge—preferably a ruler—is extremely useful.

In the **tip-to-tail method**, we “move” one of the vectors to place its “tail” onto the “tip” of the other vector. (The figure below defines the tip and tail of a vector.) Then, we draw a straight line from the tail of the first vector to the tip of the second one.



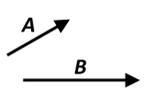
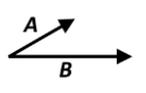
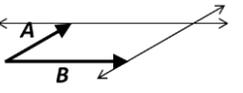
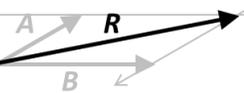
For example, to graphically add vectors **A** and **B**, with the resultant to be vector **R**:

Vectors A and B , to be added	Vector B has been moved such that its tail is at the tip of vector A	The resultant vector R is sketched, from the tail of A to the tip of B . $\mathbf{R} = \mathbf{A} + \mathbf{B}$

To show that vector addition is commutative, we can also move vector **A** to place its tail at the tip of vector **B**. The resultant is the same:

Vectors A and B , to be added	Vector A has been moved such that its tail is at the tip of vector B	The resultant vector R is sketched, from the tail of B to the tip of A . $\mathbf{R} = \mathbf{A} + \mathbf{B}$

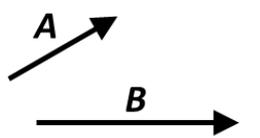
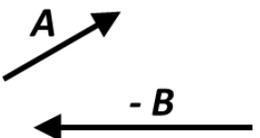
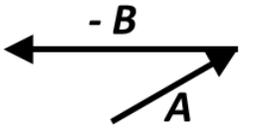
In the **parallelogram method** of graphically adding vectors, we “move” one of the vectors to place its “tail” onto the “tail” of the other vector. Then, we draw a line through the tip of the first vector, parallel to the second vector. Likewise, we draw a straight line from the tip of the second vector parallel to the first vector. Finally, we draw the resultant vector starting at the tails of the original vectors to the intersection of the lines.

<p>Vectors A and B, to be added</p>	<p>Vector B has been moved such that its tail coincides with the tail of vector A</p>	<p>A line has been drawn through the tip of A, parallel to B.</p>	<p>A line has been drawn through the tip of B, parallel to A.</p>	<p>The resultant vector has been drawn as the diagonal of the parallelogram.</p>
				

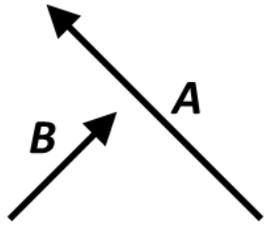
Subtraction can be thought of as adding the negative of the vector that is being subtracted. In other words:

$$\mathbf{A} - \mathbf{B} \text{ is the same as } \mathbf{A} + (-\mathbf{B})$$

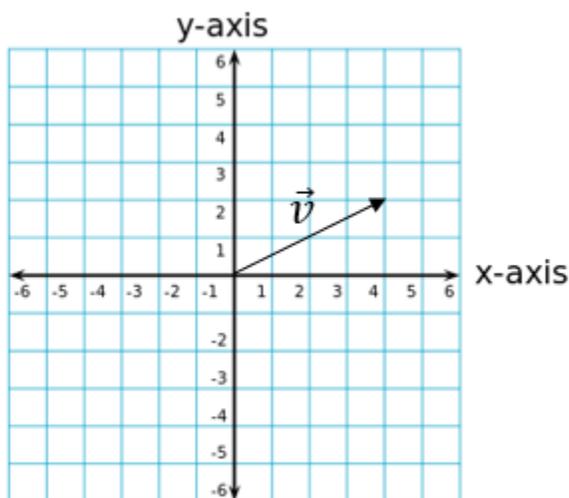
Remembering that multiplying a vector by a negative scalar (like “-1”) changes the direction of the vector by 180 degrees, we can see how to perform vector subtraction, for example, $\mathbf{R} = \mathbf{A} - \mathbf{B}$:

<p>Vectors A and B</p>	<p>Vectors A and -B</p>	<p>Lined up tip-to-tail</p>	<p>Resultant vector drawn</p>
			

Exercises: Graphically calculate the indicated sum or difference. Use of a ruler is highly encouraged. Use a colored pencil to draw in the resultant vector.

<p>Vectors A and B</p>	<p>$\mathbf{A} + \mathbf{B}$</p>	<p>$\mathbf{B} - \mathbf{A}$</p>
		

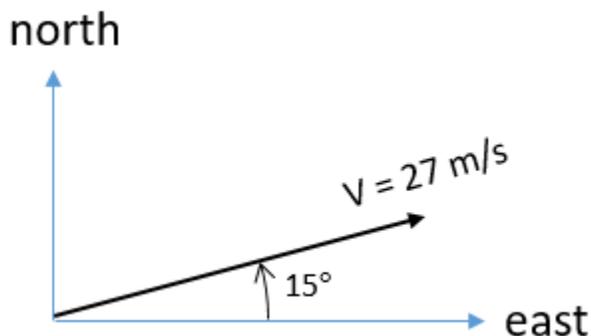
In addition to graphical methods, one can use *analytical* methods to perform vector addition and subtraction. Before learning this, however, we must learn how to **resolve** a given vector into its components—typically x- and y-components. Let us use as an example the following velocity vector, \vec{v} , drawn with its tail at the origin for convenience:



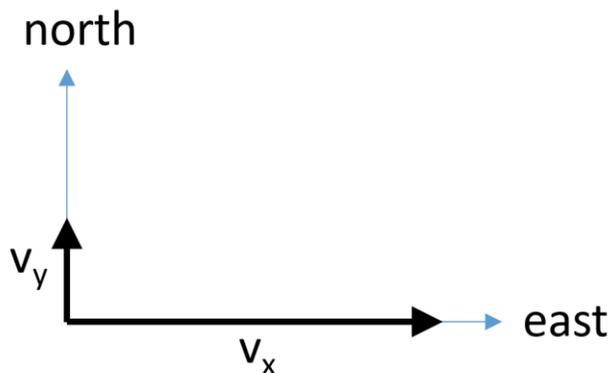
To resolve a given vector into its components means simply to determine vectors parallel to the axes such that the sum of these vectors equals the given vector. It is easy to see that, for the example above, the x-component of the velocity vector is simply 4 m/s, and the y-component is simply 2 m/s. Thus, by inspection, we can resolve the vector \vec{v} :

$$\vec{v} = 4\hat{i} + 2\hat{j} \text{ m/s}$$

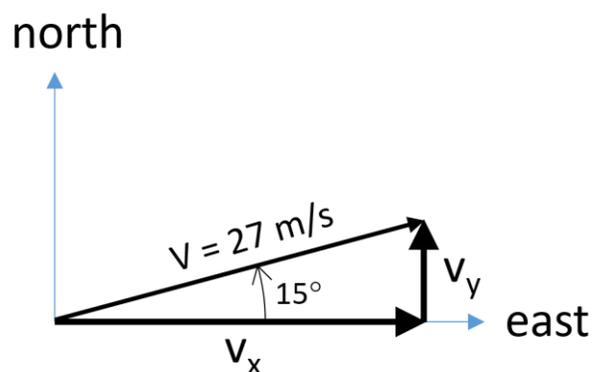
This is not so straightforward when the vector is expressed as a magnitude and a direction. For example, suppose a velocity is given as 27 m/s at a direction of 15 degrees north of east. How would one resolve this vector?



Clearly, this velocity could be the sum of two vectors \vec{v}_x and \vec{v}_y where we've taken the liberty of associating the east direction with an x-axis and the north direction with a y-axis.



The v_x and the v_y vectors are said to be the “component” vectors of the velocity vector. The act of determining the component vectors is called *resolving* the vector. We can and should use trigonometry to resolve the vector. From the following sketch, it is straightforward to see that the sine of 15 degrees equals v_y divided by 27 m/s and that the cosine of 15 degrees equals v_x divided by 27 m/s.



Using this, we can solve for the unknown quantities:

$$v_x = (27\text{m/s}) \cos(15^\circ) = 26.08 \text{ m/s}$$

$$v_y = (27\text{m/s}) \sin(15^\circ) = 6.99 \text{ m/s}$$

And finally, we can express the velocity vector in terms of its components:

$$\vec{v} = 26.08 \text{ m/s } \hat{i} + 6.99 \text{ m/s } \hat{j}$$

Generalizing, the x- and y-components of a vector \mathbf{v} that makes a counter-clockwise angle θ with respect to the positive half of the x-axis are:

$$v_x = v \cos(\theta)$$

$$v_y = v \sin(\theta)$$

And furthermore, the vector can be written in terms of unit vectors as:

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

So, getting back to the topic of adding and subtracting vectors, to do it *analytically* means to resolve each vector into its components, and then to perform the addition or subtraction on the components.

For example, to add the vector $\mathbf{A} = 275 \text{ m @}35^\circ$ to the vector $\mathbf{B} = 115 \text{ m @}215^\circ$ we need to first convert each vector to its components.

$$\vec{A} = 275 \cos(35^\circ)\text{m } \hat{i} + 275 \sin(35^\circ) \text{m } \hat{j}$$

$$\vec{A} = 225.27\text{m } \hat{i} + 157.73\text{m } \hat{j}$$

$$\vec{B} = 115 \cos(215^\circ)\text{m } \hat{i} + 115 \sin(215^\circ) \text{m } \hat{j}$$

$$\vec{B} = -94.20 \text{ m } \hat{i} - 65.96 \text{ m } \hat{j}$$

Then, perform the addition on the \hat{i} components, then on the \hat{j} components:

$$\vec{A} + \vec{B} = (225.27 - 94.20)\text{m } \hat{i} + (157.73 - 65.96)\text{m } \hat{j}$$

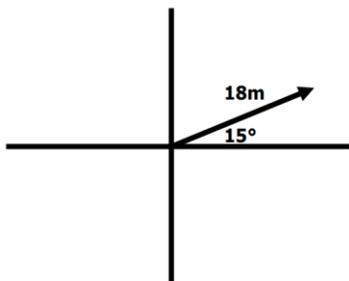
$$\vec{A} + \vec{B} = 131.07\text{m } \hat{i} + 91.77\text{m } \hat{j}$$

Although we calculated the sum of the two vectors analytically and expressed the result in rectangular coordinates, the problem might have asked for the resultant to be expressed as a magnitude and direction. In the block of space below, show that our answer is equivalent to:

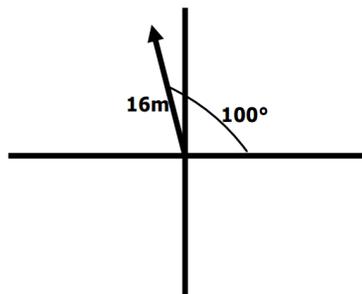
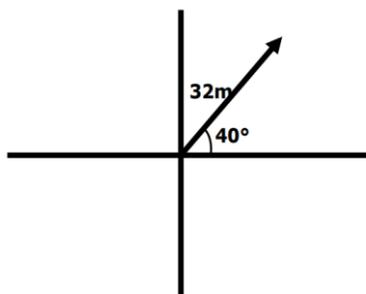
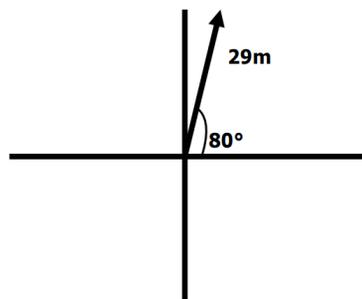
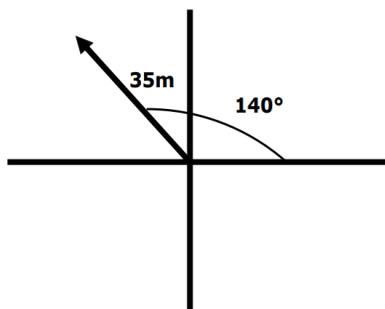
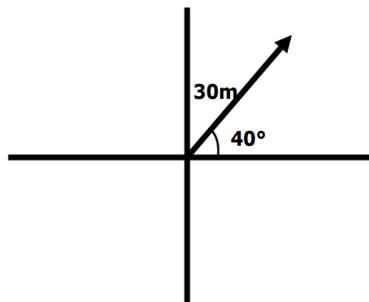
$$\vec{A} + \vec{B} = 160.00 \text{ m @}35.00^\circ$$

Vector Resolution Exercises

Resolve each vector \vec{v} into its \hat{i} and \hat{j} components. Assume the horizontal axis is x and the vertical is y. The first one has been done for you. Don't sweat the significant figures – use three or four.



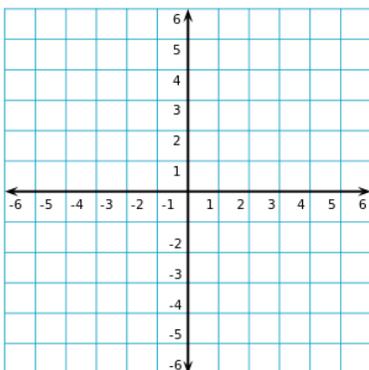
$$v_x = 18\text{m} \cos(15^\circ) = 17.39 \text{ m}$$
$$v_y = 18\text{m} \sin(15^\circ) = 4.66 \text{ m}$$
$$\vec{v} = 17.39 \text{ m } \hat{i} + 4.66 \text{ m } \hat{j}$$



Vector Construction Exercises

Convert each vector into magnitude and angle notation. A blank set of axes is provided should you need to sketch anything. The first one has been done for you.

$$\vec{v} = 17.39 \text{ m } \hat{i} + 4.66 \text{ m } \hat{j}$$

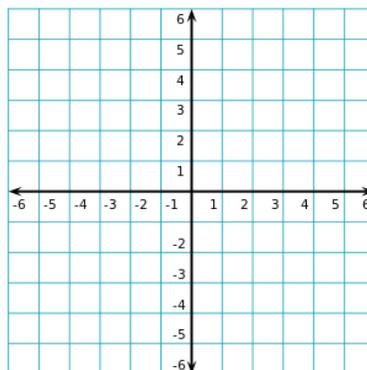


$$|\vec{v}| = \sqrt{(17.39\text{m})^2 + (4.66\text{m})^2} = 18.00\text{m}$$

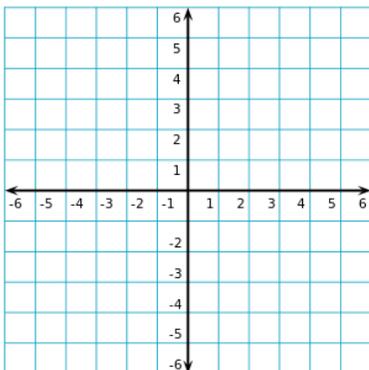
$$\theta = \tan^{-1}\left(\frac{4.66\text{m}}{17.39\text{m}}\right) = 15.00^\circ$$

$$\vec{v} = 18.00 \text{ m } @ 15.00^\circ$$

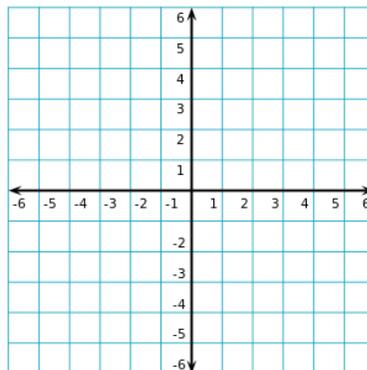
$$\vec{v} = -5.34 \text{ m } \hat{i} + 3.57 \text{ m } \hat{j}$$



$$\vec{v} = 0.07899 \text{ m } \hat{i} - 0.03455 \text{ m } \hat{j}$$



$$\vec{v} = -415 \text{ m } \hat{i} - 454 \text{ m } \hat{j}$$



Vector Addition & Subtraction Exercises

Calculate the indicated sum, expressing the resultant in either rectangular notation, or polar notation as indicated. The first one has been done for you. Use the back of this page or a piece of notebook paper for scratch paper if needed.

$\vec{A} = 17.39 \hat{i} + 4.66 \vec{j}$ $\vec{B} = -5.43 \hat{i} + 3.75 \vec{j}$ $\vec{A} + \vec{B} = \underline{11.96} \hat{i} + \underline{8.41} \hat{j}$	$\vec{A} = -17.39 \hat{i} + 4.66 \vec{j}$ $\vec{B} = 5.43 \hat{i} - 7.35 \vec{j}$ $\vec{A} + \vec{B} = \underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$	$\vec{A} = -17.39 \hat{i} + 4.66 \vec{j}$ $\vec{B} = 5.43 \hat{i} - 4.66 \vec{j}$ $\vec{A} - \vec{B} = \underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$
$\vec{A} = 13.39 \hat{i} + 4.77 \vec{j}$ $\vec{B} = -6.43 \hat{i} + 7.85 \vec{j}$ $\vec{A} + \vec{B} = \underline{\hspace{2cm}} @ \underline{\hspace{2cm}}^\circ$	$\vec{A} = -14.39 \hat{i} - 4.77 \vec{j}$ $\vec{B} = 6.43 \hat{i} + 7.35 \vec{j}$ $\vec{A} + \vec{B} = \underline{\hspace{2cm}} @ \underline{\hspace{2cm}}^\circ$	$\vec{A} = 14.39 \hat{i} + 4.77 \vec{j}$ $\vec{B} = -6.43 \hat{i} - 7.35 \vec{j}$ $\vec{A} - 2\vec{B} = \underline{\hspace{2cm}} @ \underline{\hspace{2cm}}^\circ$
$125 @ 15^\circ + 125 @ 15^\circ =$ $\underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$	$125 @ 15^\circ + 125 @ 35^\circ =$ $\underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$	$125 @ 15^\circ - 125 @ 15^\circ =$ $\underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$
$\vec{A} = 17.39 \hat{i} + 4.66 \vec{j}$ $\vec{B} = 12.34 @ 280^\circ$ $\vec{A} + \vec{B} = \underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$	$\vec{A} = 17.39 \hat{i} - 4.66 \vec{j}$ $\vec{B} = 12.34 @ 205^\circ$ $\vec{A} + \vec{B} = \underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$	$\vec{A} = 17.39 \hat{i} + 4.66 \vec{j}$ $\vec{B} = 12.34 @ 120^\circ$ $\vec{A} + \vec{B} = \underline{\hspace{2cm}} \hat{i} + \underline{\hspace{2cm}} \hat{j}$

Vector Word Problems

1. A vector \vec{D} has a magnitude of 16 meters and a direction of 0° . A vector \vec{E} has a magnitude of 12 meters and a direction of 80° . Calculate the direction and magnitude of $\vec{D} + \vec{E}$ to the nearest whole values. Show your work.

2. Two forces with magnitudes of 20 newtons and 32 newtons are applied to an object. The magnitude of the resultant force is 22 newtons. Calculate the angle between the resultant force and the 20-newton force. Show your work. Hint: This is a challenging problem. The Law of Cosines might prove helpful.

Sidebar: Third Dimensional Graphics

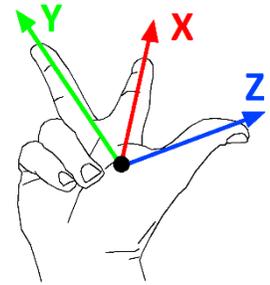
When the need arises to graphically depict a vector that is either pointing into the paper or out of the paper, there is a commonly accepted drawing practice. A vector coming out of the paper is drawn as a circle with a dot in the middle of it. This is reminiscent of an arrow with a conical tip, coming out of the paper as if shot from a bow:

Vector Coming out of Page: \odot

In a similar way, a vector going into the paper is drawn as a circle with an X in the middle of it. The X is reminiscent of the fletching of an arrow, going into the paper as if shot from a bow:

Vector Going into Page: \otimes

There is sometimes confusion regarding which way the z-axis points. The standard convention is known as the “right-hand rule” and states that if you use the index finger and thumb of your right hand to make a “gun,” and let your middle finger point in a direction that is perpendicular to the two, then the index finger points along the x-axis and the middle finger points along the y-axis and the thumb points along the z-axis. So, if you are looking at a typical x-y graph on a piece of paper, the z-axis would point up out of the paper. Try it!



Vector Multiplication

There are two kinds of vector multiplication. The first kind is called “dot product” (also called scalar product, or inner product) and yields a *scalar* result. The second kind is called “cross product” (also called vector product, or outer product) and yields a *vector* result. Which product you use will be driven by what you are trying to calculate, which will become obvious in AP Physics class.

Dot Product

The dot product of two vectors \mathbf{A} and \mathbf{B} is written as $\vec{A} \cdot \vec{B}$ (or $\mathbf{A} \cdot \mathbf{B}$) and is pronounced “A dot B.” The result of a dot product is a scalar number. Geometrically, you can think of the dot product as the projection of A onto B, times the magnitude of B. If the projection of A onto B points in the same direction as B, then the dot product is positive. If the projection of A onto B points in the opposite direction of B, then the dot product is negative.

		$\vec{A} \cdot \vec{B} = \vec{A}_b \vec{B} $
		$\vec{A} \cdot \vec{B} = - \vec{A}_b \vec{B} $

If the angle between the two vectors is known, then the dot product is easily calculated as the following:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos(\theta)$$

One can easily see that if two vectors are perpendicular to each other, then their dot product is zero. Likewise, if two vectors point in the same direction then their dot product is simply the product of their magnitudes. It should also be readily apparent that dot product is commutative, that is:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

Because the unit vectors \hat{i} , \hat{j} , and \hat{k} are mutually perpendicular, many dot-product relationships can be stated:

$\hat{i} \cdot \hat{i} = 1$	$\hat{j} \cdot \hat{j} = 1$	$\hat{k} \cdot \hat{k} = 1$
$\hat{i} \cdot (-\hat{i}) = -1$	$\hat{j} \cdot (-\hat{j}) = -1$	$\hat{k} \cdot (-\hat{k}) = -1$
$\hat{i} \cdot \hat{j} = 0$ $\hat{i} \cdot \hat{k} = 0$	$\hat{j} \cdot \hat{i} = 0$ $\hat{j} \cdot \hat{k} = 0$	$\hat{k} \cdot \hat{i} = 0$ $\hat{k} \cdot \hat{j} = 0$

When two vectors are each expressed as a magnitude and a direction, you need to perform angle subtraction in order to determine the angle between the vectors. The angle between two vectors is between 0° and 180° . For example:

$$\mathbf{A} = 23\text{m}@45^\circ \quad \mathbf{B} = 3.0\text{m}@55^\circ \quad \vec{A} \cdot \vec{B} = (23\text{m})(3.0\text{m}) \cos(55^\circ - 45^\circ) = 68 \text{ m}^2$$

Calculate the following dot products. The first one has been done for you. Don't read anything special into the raised dot in between the abbreviated units (N for newtons and m for meters) in the answer; such a raised dot is there simply to indicate multiplication of units. "N·m" is read as "newton meters."

$(17\text{N}@0^\circ) \cdot (14\text{m}@30^\circ) = 206 \text{ N}\cdot\text{m}$	$(12\text{N}@35^\circ) \cdot (14\text{m}@125^\circ) =$
$(22\text{N}@200^\circ) \cdot (19\text{m}@30^\circ) =$	$(11\text{N}@350^\circ) \cdot (14\text{m}@5^\circ) =$
$(17\text{N}@17^\circ) \cdot (14\text{m}@ - 5^\circ) =$	$(17\text{N}@ -20^\circ) \cdot (14\text{m}@-30^\circ) =$

When vectors are expressed in rectangular coordinates, calculating the dot product is even easier! Consider two 2-dimensional vectors:

$$\begin{aligned} \vec{A} &= A_x \hat{i} + A_y \hat{j} \\ \vec{B} &= B_x \hat{i} + B_y \hat{j} \end{aligned}$$

The dot product is written as:

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j}) \cdot (B_x \hat{i} + B_y \hat{j})$$

Distributing the dot operation yields:

$$\vec{A} \cdot \vec{B} = A_x \hat{i} \cdot B_x \hat{i} + A_y \hat{j} \cdot B_x \hat{i} + A_x \hat{i} \cdot B_y \hat{j} + A_y \hat{j} \cdot B_y \hat{j}$$

Recognizing that $\hat{i} \cdot \hat{i} = 1$ and that $\hat{i} \cdot \hat{j} = 0$ simplifies this dot product to:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

For example:

$$\mathbf{A} = 16.263\text{m} \hat{i} + 16.263\text{m} \hat{j}$$

$$\mathbf{B} = 1.721\text{m} \hat{i} + 2.457\text{m} \hat{j}$$

$$\vec{A} \cdot \vec{B} = (16.263\text{m})(1.721\text{m}) + (16.263\text{m})(2.457\text{m}) = 67.95 \text{ m}^2$$

This method also applies to 3-dimensional vectors as well:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Exercise: Calculate the following dot products. Show your work. The first one is already done.

$\mathbf{A} = 11\text{m} \hat{i} + 12\text{m} \hat{j}$ $\mathbf{B} = 5.0\text{m} \hat{i} + 15\text{m} \hat{j}$ $\mathbf{A} \cdot \mathbf{B} = (11\text{m})(5.0\text{m}) + (12\text{m})(15\text{m})$ $\mathbf{A} \cdot \mathbf{B} = 55\text{m}^2 + 180\text{m}^2$ $\mathbf{A} \cdot \mathbf{B} = \boxed{235\text{m}^2}$	$\mathbf{A} = 9\text{m} \hat{i} + 10\text{m} \hat{j}$ $\mathbf{B} = 6.0\text{m} \hat{i} + 12\text{m} \hat{j}$ $\mathbf{A} \cdot \mathbf{B} =$
$\mathbf{A} = -23\text{m} \hat{i} + 14\text{m} \hat{j}$ $\mathbf{B} = 19.0\text{m} \hat{i} - 12\text{m} \hat{j}$ $\mathbf{A} \cdot \mathbf{B} =$	$\mathbf{A} = -1700\text{m} \hat{i} - 1300\text{m} \hat{j}$ $\mathbf{B} = -51\text{m} \hat{i} - 15\text{m} \hat{j}$ $\mathbf{A} \cdot \mathbf{B} =$
$\mathbf{A} = 10\text{m} \hat{i} + 10\text{m} \hat{j} + 10\text{m} \hat{k}$ $\mathbf{B} = 5.0\text{m} \hat{i} + 15\text{m} \hat{j} - 25\text{m} \hat{k}$ $\mathbf{A} \cdot \mathbf{B} =$	$\mathbf{A} = 10\text{m} \hat{i} + 5.0\text{m} \hat{j}$ $\mathbf{B} = -5.0\text{m} \hat{i} + 10\text{m} \hat{j}$ $\mathbf{A} \cdot \mathbf{B} =$

The dot product can be used to determine the angle between two vectors which are expressed as components. Recall that:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

And

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos(\theta)$$

Equating the right-hand sides of these equations yields:

$$|\vec{A}| |\vec{B}| \cos(\theta) = A_x B_x + A_y B_y + A_z B_z$$

which can be manipulated to yield:

$$\cos(\theta) = \frac{A_x B_x + A_y B_y + A_z B_z}{|\vec{A}| |\vec{B}|}$$

The angle between two vectors can be as small as 0° or as large as 180° ; thankfully the inverse cosine operation is unambiguous over this interval. Therefore, the angle between two vectors can be determined by:

$$\theta = \cos^{-1} \left(\frac{A_x B_x + A_y B_y + A_z B_z}{|\vec{A}| |\vec{B}|} \right)$$

Keep in mind the numerator of the argument of this cosine function might be positive or negative, but the denominator will always be positive.

Challenge Problem:

Calculate the angle between the following pair of 3-dimensional vectors. As always, show your work.

$$\mathbf{A} = 9 \hat{i} + 8 \hat{j} + 7 \hat{k}$$

$$\mathbf{B} = 4 \hat{i} - 5 \hat{j} + 6 \hat{k}$$

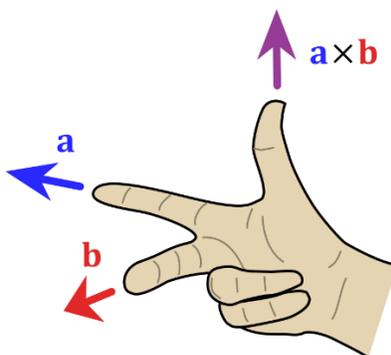
Cross Product

The cross product of two vectors \mathbf{A} and \mathbf{B} is written as $\vec{A} \times \vec{B}$ (or $\mathbf{A} \times \mathbf{B}$) and is pronounced “A cross B.” The result of a cross product is a vector which is perpendicular to the two vectors being crossed.

The magnitude of a cross product is the magnitude of the first vector times the magnitude of the second vector, times the sine of the angle between them.

$$\text{If } \vec{C} = \vec{A} \times \vec{B} \text{ then } |\vec{C}| = |\vec{A}||\vec{B}|\sin(\phi).$$

The direction of the cross product is given by using the right hand in the following way. If the direction of the first vector is along the index finger and the direction of the second vector is along the middle finger, then the direction of the cross product vector is along the direction of the thumb. Note that it is not required for vectors \mathbf{A} and \mathbf{B} to be perpendicular to each other, however \mathbf{C} will be perpendicular to both \mathbf{A} and \mathbf{B} .

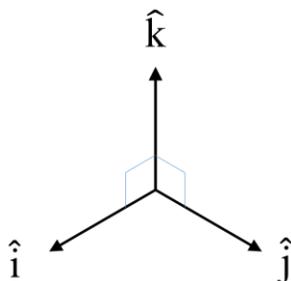


One can easily see that if two vectors are perpendicular to each other ($\phi = 90^\circ$), then the magnitude of their cross product is just the product of their magnitudes. Likewise, if two vectors point in the same direction then their cross product is zero. It should also be readily apparent that cross product is not commutative, in fact:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Because the unit vectors \hat{i} , \hat{j} , and \hat{k} are mutually perpendicular, many cross-product relationships can be stated:

$\hat{i} \times \hat{i} = 0$	$\hat{j} \times \hat{j} = 0$	$\hat{k} \times \hat{k} = 0$
$\hat{i} \times (-\hat{i}) = 0$	$\hat{j} \times (-\hat{j}) = 0$	$\hat{k} \times (-\hat{k}) = 0$
$\hat{i} \times \hat{j} = \hat{k}$ $\hat{i} \times \hat{k} = -\hat{j}$	$\hat{j} \times \hat{i} = -\hat{k}$ $\hat{j} \times \hat{k} = \hat{i}$	$\hat{k} \times \hat{i} = \hat{j}$ $\hat{k} \times \hat{j} = -\hat{i}$



When two vectors in the x-y plane are each expressed as a magnitude and a direction, you can perform angle subtraction in order to determine the angle between the vectors. For example:

$$\mathbf{A} = 23\text{m}@45^\circ \quad \mathbf{B} = 3.0\text{m}@55^\circ \quad |\vec{A} \times \vec{B}| = (23\text{m})(3.0\text{m}) \sin(55^\circ - 45^\circ) = 12 \text{ m}^2$$

When two vectors \mathbf{A} and \mathbf{B} lie in the x-y plane, the cross product will point either in the +z or the -z direction. To figure out which it is for the previous example, align the index finger of your right hand with the angle of the first vector (45°) and the middle finger with the angle of the second vector (55°), remembering that positive angles are measured counterclockwise from the +x axis. Your thumb should then be pointing up out of the paper, which is the positive z direction. Therefore, we express the cross product using unit vector notation:

$$\vec{A} \times \vec{B} = 12 \text{ m}^2 \hat{k}$$

Worth repeating is that if two vectors lie in the x-y plane—like most of the problems we’ll be solving in physics—then their cross product will always point in a direction perpendicular to the x-y plane.

Another way to perform a cross product is to express the vectors in rectangular coordinates and use the distributive property. For example, the previous two vectors can be expressed as:

$$\vec{A} = 23 \cos(45^\circ) \hat{i} + 23 \sin(45^\circ) \hat{j} \text{ m}$$

$$\vec{B} = 3.0 \cos(55^\circ) \hat{i} + 3.0 \sin(55^\circ) \hat{j} \text{ m}$$

The cross product $\vec{A} \times \vec{B}$ can then be written out as such:

$$\vec{A} \times \vec{B} = (23 \cos(45^\circ) \hat{i} + 23 \sin(45^\circ) \hat{j}) \times (3.0 \cos(55^\circ) \hat{i} + 3.0 \sin(55^\circ) \hat{j}) \text{ m}^2$$

One can then use the distributive property, being careful to preserve the order of the products:

$$\begin{aligned} \vec{A} \times \vec{B} = & [23 \cos(45^\circ) \hat{i} \times 3.0 \cos(55^\circ) \hat{i}] + [23 \sin(45^\circ) \hat{j} \times 3.0 \cos(55^\circ) \hat{i}] + [23 \cos(45^\circ) \hat{i} \times 3.0 \sin(55^\circ) \hat{j}] \\ & + [23 \sin(45^\circ) \hat{j} \times 3.0 \sin(55^\circ) \hat{j}] \end{aligned}$$

Any bracketed term that involves a unit vector crossed with itself vanishes, so we have:

$$\vec{A} \times \vec{B} = [0] + [23 \sin(45^\circ) \hat{j} \times 3.0 \cos(55^\circ) \hat{i}] + [23 \cos(45^\circ) \hat{i} \times 3.0 \sin(55^\circ) \hat{j}] + [0]$$

And now applying that $\hat{j} \times \hat{i} = -\hat{k}$ and that $\hat{i} \times \hat{j} = \hat{k}$ we have:

$$\vec{A} \times \vec{B} = [0] + [-23 \sin(45^\circ) 3.0 \cos(55^\circ) \hat{k}] + [23 \cos(45^\circ) 3.0 \sin(55^\circ) \hat{k}] + [0]$$

$$\vec{A} \times \vec{B} = [0] + [-27.98 \hat{k}] + [39.97 \hat{k}] + [0]$$

$$\vec{A} \times \vec{B} = 12 \text{ m}^2 \hat{k}$$

A third way to calculate the cross product—which is extremely useful if the vectors are already in rectangular coordinates—is to use the “determinant method.” In the determinant method, a three-by-three matrix is constructed with the unit vectors comprising the first row and the components of the two vectors comprising the second and third row. The cross product is simply the determinant of the matrix! **(If it’s been a while since you’ve calculated determinants, it might be good to brush up on them.)**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_y B_z - A_z B_y) - \hat{j}(A_x B_z - A_z B_x) + \hat{k}(A_x B_y - A_y B_x)$$

This method is especially useful for calculating the cross product of two vectors that do not lie in the x-y plane.

For example, let $\vec{A} = 9.0\hat{i} + 11\hat{j} + 13\hat{k}$ and $\vec{B} = -5.0\hat{i} + 3.0\hat{j} - 2.0\hat{k}$. Calculate the cross product $\vec{A} \times \vec{B}$ of the vectors AND determine the angle between the vectors.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 9.0 & 11 & 13 \\ -5.0 & 3.0 & -2.0 \end{vmatrix} = \hat{i}(-22 - 39) - \hat{j}(-18 + 65) + \hat{k}(27 + 55)$$

$$\vec{A} \times \vec{B} = -61\hat{i} - 47\hat{j} + 82\hat{k}$$

To calculate the angle between the vectors, you *might* be tempted use the relation:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin(\phi)$$

where the left-hand side is calculated as the magnitude of the vector calculated using the determinant method. You *might* be tempted to re-arrange this equation to yield:

$$\sin(\phi) = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|}$$

However, you should NOT use the inverse sine operation on the right-hand-side because the inverse sine is NOT, repeat NOT, unambiguous over the interval 0° to 180° . (Calculate the sine of 120° to get $\sqrt{3}/2$ then take the inverse sine of $\sqrt{3}/2$ to see if you get 120° . See what I mean?)

Instead, you should use the dot product to calculate the angle between two vectors. So, for the above example,

$$\vec{A} \cdot \vec{B} = (9.0)(-5.0) + (11)(3.0) + (13)(-2.0) = -38$$

$$|\vec{A}| = \sqrt{9.0^2 + 11^2 + 13^2} = 19.26$$

$$|\vec{B}| = \sqrt{5.0^2 + 3.0^2 + 2.0^2} = 6.16$$

$$\phi = \cos^{-1}\left(\frac{-38}{(19.26)(6.16)}\right) = 109^\circ$$

Student exercise:

Let $\vec{A} = 2.0\hat{i} + 3.0\hat{j} + 1.0\hat{k}$ and $\vec{B} = -7.0\hat{i} + 6.0\hat{j} - 5.0\hat{k}$. Calculate the cross product $\vec{A} \times \vec{B}$ of the vectors AND determine the angle between the vectors.

Student exercise, continued